110.202 – Calculus III 2/6/2017

Section 1.2

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1.2 Inner Product, Length, Distance

Inner Product

Suppose we have two vectors $\vec{a}, \vec{b}$ in $\mathbb{R}^3$, and we want to understand the angle between them.

Definition 1 *If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then the inner product of $\vec{a}$ and $\vec{b}$, denoted $\vec{a} \cdot \vec{b}$, is equal to $a_1b_1 + a_2b_2 + a_3b_3$.*

Note that the inner product is a scalar!
The following properties are true of the inner product

- $a \cdot a \geq 0$, and $a \cdot a = 0$ if and only if $a = 0$
- For a scalar c, $c\vec{a} \cdot \vec{b} = \vec{a} \cdot c\vec{b} = c(\vec{a} \cdot \vec{b})$.
- For three vectors $\vec{a}, \vec{b}, \vec{c}$, $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

By the Pythagorean theorem, the length of a vector $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ is equal to $\sqrt{a_1^2 + a_2^2 + a_3^2}$. We denote this by $\|\vec{a}\|$ and call it the norm of $\vec{a}$. Note that $\|\vec{a}\| = (\vec{a} \cdot \vec{a})^{1/2}$.

Unit Vectors

Units with norm 1 are called unit vectors. Note that $\vec{i}, \vec{j}, \vec{k}$ are unit vectors. We can transform any vector $\vec{a}$ into a unit vector by dividing it by $\|a\|$. The vector $\vec{a}/\|\vec{a}\|$ is known as a normalized vector.

Angle Between Vectors

Let a, b be two vectors in $\mathbb{R}^3$, and let $\theta \in [0, \pi]$ be the angle between them. Then $a \cdot b = \|a\|\|b\| \cos\theta$. This fact follows from application of the law of cosines.

Inequalities

Cauchy-Schwarz:
For any two vectors a, b, we have

$$|a \cdot b| \leq \|a\|\|b\|,$$

and equality holds when a is a scalar multiple of b or one of them is zero.
Triangle Inequality:
For any two vectors $a, b,$, we have $\|a + b\| \leq \|a\| + \|b\|$.

Orthogonal Projection

The projection of a vector u onto a vector v is a vector in the direction of v, and, specifically, is the vector $\text{proj}_v(u) = \frac{u \cdot v}{\|v\|^2} v$.