1.3 Matrices, Determinants, and the Cross Product

2 × 2 Matrices

We define a 2×2 matrix to be an array of scalars $a_{11}, a_{12}, a_{21}, a_{22}$ as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The determinant of such a matrix is the scalar value $a_{11}a_{22} - a_{21}a_{12}$.

3 × 3 Matrices

3×3 matrices are similarly defined,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Determinants of 3×3 matrices are more complex, but there's a small trick to calculate them faster – the book has an explicit formula but (what I showed in section) is easier to remember.

Properties of Determinants

If we have a matrix, we can perform so-called row or column operations and these affect the determinant in predictable ways.

If we switch two rows or columns, the sign of the determinant is changed. If we multiply any row by a scalar k, the determinant is also multiplied by k(this also means if a row or column is all zeroes then the determinant is zero). The last fundamental property is that if we add a row to another (or column to another), the determinant remains unchanged.

Example 1 *So, what happens if some row is a linear combination of another? Let's do an example, then a proof.*

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 3 & 3 \end{vmatrix}$$

Here, the third row is the difference of the second and first rows. Check det is 0.

Example 2 *In general,*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \alpha a_{11} + \beta a_{21} & \alpha a_{12} + \beta a_{22} & \alpha a_{13} + \beta a_{23} \end{vmatrix}$$

Here, the third row is a general linear combination of the first two. Check det is 0.

We can also "expand" the determinant of a 3×3 matrix along minors across any row or column. We just have to remember to follow a checkerboard pattern. Verify with Ex1.

Cross Products

Suppose $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$. Then $\vec{a} \times \vec{b}$ is the cross product of $\vec{a}$ and $\vec{b}$ and is defined to be the vector

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

For cross products,

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. Note that this means $\vec{a} \times \vec{a} = -\vec{a} \times \vec{a}$, so $\vec{a} \times \vec{a} = \vec{0}$. Furthermore, note that $\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}$, and $\vec{k} \times \vec{i} = \vec{j}$.

The length of the cross product vector is $\|\vec{a}\times\vec{b}\|^2 = \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b}^2) = \|\vec{a}\|^2\|\vec{b}\|^2 - \|a\|^2\|b\|^2\cos^2\theta = \|a\|^2\|b\|^2\sin^2\theta$.

Geometry of determinants

The absolute value of the determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is the area of the parallelogram whose sides are (a_1, a_2) and (b_1, b_2).

For 3×3 determinants, the absolute value of the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is the volume of the solid with sides $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)$.

Equation of Planes in Space

The equation of the plane through the point (x_0, y_0, z_0) with a normal vector $\vec{n} = (A, B, C)$ is $\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$.

This is because the plane consists of all points (x, y, z) such that the vector $(x, y, z) - (x_0, y_0, z_0)$ is perpendicular to $\vec{n}$.

Related to this is the concept of distance from point to the plane. If we have a plane through (x_0, y_0, z_0) with unit normal vector $\vec{n}$, the distance from a point (x_1, y_1, z_1) to the plane is the orthogonal projection of the vector $\vec{v} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$ onto $\vec{n}$. Hence the distance is just $|\vec{v} \cdot \vec{n}|$. There is a more complicated formula in the book, but it's often easier to understand where it comes from. Just make sure that when you apply this formula that $\vec{n}$ is a unit vector. Otherwise it will not apply!