110.202 – Calculus III 12/08/2016

Final Exam Practice Problems

Section: 3 (David Li)

Most of these problems are adapted from final exams from MAT203 (Advanced Multivariable Calculus) at Princeton University.

1. **F06-07 #1**

$$f(x,y) = x^2 - 4x + y^2 - 4y + 10$$

   (a) Find the critical points of f.

   (b) Find the maximum and minimum of f subject to $x^2 + y^2 = 25$.

   (c) Find the maximum and minimum of f subject to $x^2 + y^2 \le a$.
   Explain how the answer depends on the value of a.

2. **F02-03 #1** Find and classify the critical points of $z = (x^2 - y^2)e^{(-x^2-y^2)/2}$.

3. **S05-06 #6** Find the local and global extrema and their function values of the function $f(x,y,z) = xy + yz + xz$ in the closed unit ball $B = \{(x,y,z) : x^2 + y^2 + z^2 \le 1\}$.

4. **F05-06 #1** Let γ be the curve in the xy-plane defined by

$$1 = \sqrt{x} + \sqrt{y/8},\ x, y \ge 0.$$

   (a) Sketch the curve.

   (b) Compute the tangent line at the point $(\frac{1}{4}, 2)$.

   (c) Find the distance from γ to the origin.

5. **S04-05 #2** Find the point on the surface $z^2 - xy = 1$ nearest to the origin.

6. **S02-03 #2** Use Lagrange multipliers to show that if $\sum_{i=1}^{n} x_i^2 = 1$ then $x_1^2 x_2^2 \cdots x_n^2 \le 1/n$.
   Conclude that the geometric mean of n positive integers is always less than or equal to the arithmetic mean: i.e. if $a_1, a_2, \ldots, a_n > 0$, then

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

7. Compute the limits (if they exist)

   (a) **F07-08 #2**

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2 + x^2 y}{x^4 - y^2 + xy^2}, \lim_{(x,y)\to(0,0)} \frac{e^{-y^2} - e^{x^2}}{x^2 + y^2}.$$

   (b) **F06-07 #3b**

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + xy^2}{x^2 + y^2}, \lim_{(x,y)\to(0,0)} \frac{\log(1 + x^2 + y^2)}{x^2 + y^2}.$$

8. **F03-04 #2** Evaluate the integral

$$\iiint_{\mathbb{R}^3} \frac{dxdydz}{[1+(x^2+y^2+z^2)^{3/2}]^{3/2}}.$$

9. **S07-08 #5** Let $\mathbf{F}(x,y,z) = -10\mathbf{r}/r^3$ be a vector field on $\mathbb{R}^3 - (0,0,0)$ where $\mathbf{r} = (x,y,z)$ and $r(x,y,z) = \sqrt{x^2+y^2+z^2}$.

   (a) Calculate $\text{div}(\mathbf{F})$.

   (b) Is $\mathbf{F} = \text{curl}(\mathbf{G})$ for some vector field $\mathbf{G}$ defined on $\mathbb{R}^3 - (0,0,0)$?

10. **F03-04 #3b** For the following integral, sketch the region of integration, interchange the order, and evaluate the integral in the new order.

$$\int_1^4 \int_1^{\sqrt{x}} x^2 + y^2 \, dydx.$$

11. **F07-08 #8**

   (a) The Laplace operator ∇^2 on $\mathbb{R}^3$ is defined by

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

   in Cartesian coordinates. Write ∇^2 in terms of cylindrical coordinates (i.e. in terms of partial derivatives with respect to r, θ, and z.)

   (b) (Also **S05-06 #5**) A C^2 function is said to be harmonic if $\nabla^2 f(x,y,z) = 0$ for all x, y, z. Suppose f and g are harmonic functions. Show that

$$\int_S (f\nabla g - g\nabla f) \cdot d\mathbf{S} = 0,$$

   for any sphere S (oriented however you wish) in $\mathbb{R}^3$.

12. **S05-06 #4** Let $T(u,v) = (u^2 - v^2, 2uv)$ and let D^* be the region given by $u \geq 0, v \geq 0, u^2 + v^2 \leq 1$. Let D be the image of D^* under T.

   (a) Show that T is a one-to-one function from D^* to D.

   (b) Compute

$$\iint_D (x^2+y^2)^{-1/2} \, dxdy.$$

13. **F02-03 #4**

   (a) Find a linear transformation taking the square $[0,1] \times [0,1]$ to the parallelogram P with vertices at $(0,0), (-1,2), (3,1), (2,3)$.

   (b) Compute

$$\iint_P (x^2+y^3) \, dxdy.$$

14. **S03-04 #1**

    (a) Let $\Phi(u,v) = (2u - v, u + v)$. What is the image of Φ under the set D^*, where D^* is the unit square $[0,1] \times [0,1]$?

    (b) Compute $\iint_D x^2 - y \, dydx$ where D is the answer to part (a).

15. **F01-02 #5** If $\mathbf{F} = 2xy^2 e^{x^2} \, \mathbf{i} + 2ye^{x^2} \, \mathbf{j} + z \, \mathbf{k}$, and C is parameterized by $\mathbf{c} : [0,1] \mapsto \mathbb{R}^3$ defined by $\mathbf{c}(t) = (\cos(\pi t) - 1, \sin(\pi t), t^2 + t + 1)$, compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{S}$.

16. **S04-05 #4** Find the area of the surface defined by $z = x^2 + 2y^2, x^2 + 4y^2 \leq 1$.

17. **F01-02 #7** Let W be the region in the first octant bounded by the planes $y = 0, z = 0, x = y$ and by the unit sphere $x^2 + y^2 + z^2 = 1$.

    (a) Compute
    $$\iiint_W e^{-(x^2+y^2+z^2)^{3/2}} \, dV.$$

    (b) If S is the boundary of W and $\mathbf{F} = (3x - z^4)\mathbf{i} - (x^2 - y) \, \mathbf{j} + xy^2 \, \mathbf{k}$, compute the surface integral
    $$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

18. **S03-04 #3** Let C be the triangle with vertices at $(0,0), (1,0), (0,1)$ oriented in a counter-clockwise direction. Evaluate
    $$\int_C \frac{e^x}{e^x + y^2} \, dx + \left(x + \frac{2y}{e^x + y^2} \right) \, dy.$$

19. **F01-02 #6** Let C be the boundary of the rectangle with sides $x = 1, y = 2, x = 3, y = 3$. Evaluate
    $$\int_C \frac{2y + \sin x}{1 + x^2} \, dx + \frac{x + e^y}{1 + y^2} \, dy.$$

20. **S02-03 #5** Let K be the closed unit disc defined by $x^2 + y^2 \leq 1$, and let D be the region outside K which is bounded on the left by $y^2 = 2(x + 2)$ and on the right by $x = 2$. Let ∂D be the boundary of D. Evaluate
    $$\int_{\partial D} \left(-\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy \right).$$

21. **S02-03 #6** Let S be the unit sphere defined by $x^2 + y^2 + z^2 = 1$, and let $\mathbf{F}(x,y,z) = e^{x^2 + \sin(z)} \, \mathbf{i} + \sinh(\sin(y) + z) \, \mathbf{j} + z \, \mathbf{k}$. Evaluate the surface integral
    $$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

22. **S04-05 #7** Let Σ be the portion of the sphere $x^2 + y^2 + z^2 = 4$ which lies above the xy-plane, oriented with upward normal. Find the flux of $\mathbf{F} = (x^3 + y^2 \cos z, y^3 + e^x \sin z, z^3 - 1)$ across Σ.

23. **F05-06 #7** Let S be the surface parameterized by the function

$$T(\theta, \phi) = ((3 + \cos(\phi))\cos(\theta), (3 + \cos(\phi))\sin(\theta), \sin(\phi)),$$

where $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$.

(a) Compute the tangent plane P of S at the point $T(0, \frac{\pi}{4})$.

(b) Find all points on the surface S with tangent plane parallel to P.

(c) If S has the orientation induced by the parameterization T, and $\mathbf{F}(x, y, z) = \left(xy^2, x\sinh(x)e^x - \frac{y^3}{3} + y, 2 - z\right)$, compute

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

24. **S03-04 #4** Let Σ be the portion of the surface $z = 4 - (x^2 + y^2 - 1)^2$ which lies above the xy-plane, oriented with an upward-pointing normal vector. Use Gauss' Theorem to evaluate

$$\iint_\Sigma (e^y \cos z\mathbf{i} + (y + e^x \cos z)\mathbf{j} + \mathbf{k}) \cdot d\mathbf{S}.$$

25. Let $\mathbf{F}(x, y, z) = (x(1 - x^{980}y^{1000}z^{2002}), y + e^{-yx^{482}}, z)$, where Ω is a solid region in $\mathbb{R}^3$. Suppose the flux of the vector field $\mathbf{F}$ outward across $\partial\Omega$ is greater than or equal to 12. Show that Volume $(\Omega) \geq 4$.

26. **S01-02 #3** For what A, B is the vector field

$$\mathbf{F} = (2A + B)e^x\ \mathbf{i} + (Ae^x + z\cos y)\ \mathbf{j} + B\sin y\ \mathbf{k}$$

conservative? Find the potential function when it exists.

27. **F06-07 #7**

(a) Is the vector field

$$\mathbf{F} = (e^x \sin x + 3x^2yz, x^3z + \tan z, e^x \cos z + x^3y + y\sec^2 z)$$

conservative? If so, find the potential function. Then compute

$$\int_C \mathbf{F} \cdot d\mathbf{s},$$

where C is the curve parameterized by $\mathbf{c}(t) = (\frac{4\sqrt{2}}{\pi}\cos t, \sqrt{2}\sin t, t)$ for $0 \leq t \leq \pi/4$.

(b) Consider two vector fields $\mathbf{F}$ and $\mathbf{G}$. Answer the following questions, with justification.

i. If $\mathbf{F}$ and $\mathbf{G}$ are both not conservative, is it possible that $\mathbf{F} - \mathbf{G}$ is conservative?

ii. If $\mathbf{F}$ and $\mathbf{G}$ are both conservative, is it possible that $\mathbf{F} - \mathbf{G}$ is not conservative?

iii. If $\mathbf{F}$ is conservative but $\mathbf{G}$ is not, is it possible that $\mathbf{F} - \mathbf{G}$ is conservative?

iv. Assuming $\mathbf{F} = \nabla f$ in $\mathbb{R}^3$ for f at least class C^1, and consider a curve C parameterized by $\mathbf{c}(t)$ satisfying $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$ (a flow line). Can it occur that C lies on a level surface of f? If yes, when?

28. **S07-08 # 3** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(u, v) = (5u - v, 2u + 3v)$. Let D be the region $0 \leq u \leq 1, 0 \leq v \leq 2$ and let $R = T(D)$.

(a) Sketch R.

(b) Compute

$$\iint_R (x^2 + y)\, dxdy.$$