110.202 – Calculus III February 28, 2017

Section Midterm #1 Practice Problems

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These problems are from Multivariable Calculus exams from Princeton University 2001-2007. I distribute these problems for your practice – I don't have the answers to the questions on hand, but if you ask me to explain the solution (over email, office hours, Math Help Room). If asking over email, I'll give you a sketch of how to approach the problem.

1. **[Final Exam] F07-08 #2**

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2 + x^2y}{x^4 - y^2 + xy^2}, \lim_{(x,y)\to(0,0)} \frac{e^{-y^2} - e^{x^2}}{x^2 + y^2}.$$

2. **[Final Exam] F06-07 #3b**

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + xy^2}{x^2 + y^2}, \lim_{(x,y)\to(0,0)} \frac{\log(1 + x^2 + y^2)}{x^2 + y^2}.$$

(*10 points*). Consider the line of intersection of surfaces $z = x^2 + 3y^2$ and $z = 4 - 3x^2 - y^2$. Write the equation of the tangent line to it at their common point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2)$.

1. (*15 points*). Find all the points on the hyperbolic paraboloid $z = x^2 - 3y^2$ at which the tangent plane is parallel to the plane $8x + 3y - z = 4$.

2. (*15 points*). Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1-function and $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $g(r, \phi) = (r\cos\phi, r\sin\phi)$. Consider the composition $h(r,\phi) = f \circ g(r,\phi) = f(g(r,\phi)$. Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at point $(x, y) = g(1, \pi/2) = (0, 1)$ provided that $\frac{\partial h}{\partial r}(0,1) = -3$ and $\frac{\partial h}{\partial \phi} = 1$.

3. (*10 points*). Consider the path $\mathbf{c}(t) = (e^t, \sqrt{2}t, e^{-t})$, $0 \le t \le 1$ in $\mathbb{R}^3$.
   a. Find the velocity and acceleration vectors at $t = 1/3$.
   b. Write the equation of the tangent line to this path at $\mathbf{c}(1/3)$.
   c. Compute the length of this path.

1. *[15 points]*

(a) Let $\mathbf{p}(t)$ be a path and suppose that $\|\mathbf{p}(t)\| = 1$ for all t. Show that $\mathbf{p}'(t)$ is always perpendicular to $\mathbf{p}(t)$.

(b) Let $\mathbf{q}(t)$ be the unit vector parallel to $\mathbf{p}'(t)$ and let $\mathbf{r}(t) = \mathbf{p}(t) \times \mathbf{q}(t)$. Show that $\mathbf{r}'(t) = \mathbf{p}(t) \times \mathbf{q}'(t)$.

(c) Deduce that $\mathbf{r}'(t)$ is a scalar multiple of $\mathbf{p}'(t)$.

2. *[15 points]* Consider the two vectors $\mathbf{v} = \mathbf{i} + 3\mathbf{k}$ and $\mathbf{w} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. Find

(a) $\|2\mathbf{v} + 3\mathbf{w}\|$

(b) $\mathbf{v} \cdot \mathbf{w}$

(c) $\mathbf{v} \times \mathbf{w}$

3. *[15 points]* Let P_1 be the plane $x + y = 3$ in $\mathbb{R}^3$ and let P_2 be the plane $x + 2z = 0$ in $\mathbb{R}^3$.

(a) Find a parametric equation of the line L obtained by intersecting P_1 with P_2.

(b) Find the distance between the point $(-4, 0, 1)$ and the line L.

4. *[15 points]* A rocket in space follows the path $\mathbf{c}(t) = (2e^t, t, e^{2t})$.

(a) Find the arc length of the rocket's path between the point $(2, 0, 1)$ and the point $(2e, 1, e^2)$.

(b) At time $t = 2$, the rocket's motor is turned off and the rocket is allowed to drift in a straight line. Find the position of the rocket at time $t = 4$.

Q3. (a) Find an equation for the plane tangent at $(0, 2, 1/4)$ to the graph of the function $f(x, y) = \frac{e^x}{x^2+y^2}$

(b) Let $g(x, y, z) = x^2 + y^2 - 4z^2$. Find an equation of the plane tangent at $(0, 2, 1/4)$ to a level surface of g.

(c) Describe parametrically the line which is tangent at $(0, 2, 1/4)$ to the curve of intersection between the graph of f and the level surface of g.

1. *(10 points)* Consider the points

$$P = (1/2, 1, 1) \quad Q = (1, 1, 0) \quad R = (7/2, 3, 2)$$

(a) Find a vector $\mathbf{n}$ perpendicular to both PQ and PR.

(b) Find the equation of the plane through P, Q and R.

(c) Find the area of the triangle PQR.

(d) Find a parametric equation for the line L through Q and parallel to PR.

(e) Find the angle between the line L and the line PQ.

2. *(10 points)* Consider the trajectory given by

$$\mathbf{r}(t) = (e^t + e^{-t}, e^t - e^{-t}, 2t)$$

(a) Compute the velocity and the unit tangent vector at every point.

(b) Find the length of the curve from $t = 0$ to $t = 2$.

3. *(10 points)* Find the limits below, or show that they do not exist.

(a) $$\lim_{(x,y)\to(0,1)} \frac{1-x^2}{\sqrt{(x-1)^2+y^2}\sqrt{(x+1)^2+y^2}}$$

(b) $$\lim_{(x,y)\to(-1,0)} \frac{1-x^2}{\sqrt{(x-1)^2+y^2}\sqrt{(x+1)^2+y^2}}$$

1. *(10 points)* Find a parametric equation for the line that contains the point $(1, 1, -2)$ and intersects the line $x = -3 + 2t$, $y = 4 + t$, $z = 2 + 3t$ at a right angle.

2. *(10 points)* Consider the function $f(x, y) = x^2 - 2x + y^2$.

a). Sketch and describe in words the surface $z = f(x, y)$.

b). Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 1, 0)$.

c). What is the linear approximation of $f(x, y)$ at the point $(1, 1)$?

2. *(10 points)* A spaceship is moving on the trajectory given by:

$$r(t) = (t \cdot \cos(\ln t), t \cdot \sin(\ln t), t), \quad 5 \leq t \leq 10.$$

a. Compute the velocity and the unit tangent vector for every time between $t = 5, t = 10$.

b. Find the length of its path between $t = 5, t = 10$.

c. At $t = 10$ the spaceship turns off its engines and continues moving with constant velocity on a straight line. Where will it be at $t = 20$?

3. *(10 points)* Compute the following limits or show that they do not exist.

(a) $$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^4}.$$

(b) $$\lim_{(x,y)\to(1,1)} \frac{\sin(\pi x) + \sin(\pi y)}{|x - 1| + |y - 1|}$$

(c) $$\lim_{(x,y)\to(0,0)} \frac{\cos x \cos y}{e^x + e^y}.$$

l. *(20 points)* Let $c(t) = (\cos t, t^2 - 2t, \sin t)$ and $d(t) = (3 \sin t, t^2 + t, -3 \cos t)$.

(a) At what times, are the velocity vectors of c(t) and d(t) perpendicular?

(b) At what times, are the speed of c(t) and d(t) equal?

(c) Suppose that $f(x, y, z)$ be a function whose gradient at $(-1, \pi^2 - 2\pi, 0)$ is given by $6\vec{i} + \vec{j} - 9\vec{k}$. Find the derivative of $f(c(t))$ at $t = \pi$.