110.202 – Calculus III April 4, 2017

Section Midterm #2 Practice Problems

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These problems are from Multivariable Calculus exams from Princeton University 2001-2007. I distribute these problems for your practice – I don't have the answers to the questions on hand, but if you ask me to explain the solution (over email, office hours, Math Help Room), I'll work it out with you. If asking over email, I'll give you a sketch of how to approach the problem.

Note that these problems aren't necessarily representative of the actual exam questions – but I think they will be helpful in your preparation.

Q2. (30pts.) Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 - x - y + 1$ on the unit disc $x^2 + y^2 \leq 1$.

Problem 6. Find the local and global maxima and minima of the function $f(x, y, z) = xy + yz + zx$ in the closed unit ball $B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$. Find both the locations of the extrema and the minimum and maximum values that f takes at these locations.

2. (20 points) Find the point on the surface $z^2 - xy = 1$ which is nearest to the origin.

Q3. (20pts.) Fine the point(s) on the surface $z^2 - 10xy = 10$ nearest to the origin.

Problem 2. Let $f(x, y)$ be a C^2 scalar-valued function defined on all of $\mathbb{R}^2$ with $\nabla^2 f = 0$, i.e., $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ for all x, y. Let $x = r\cos\theta$ and $y = r\sin\theta$. Express $\frac{\partial^2 f}{\partial \theta^2}$ in terms of $\frac{\partial^2 f}{\partial r^2}$, $\frac{\partial f}{\partial r}$, and r.

Q5. (20pts.) Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfy

$$g(0,0) = 0 \qquad \frac{\partial^2 g}{\partial u^2}(0,0) = 6$$

$$\frac{\partial g}{\partial u}(0,0) = 2 \qquad \frac{\partial^2 g}{\partial v^2}(0,0) = 4$$

$$\frac{\partial g}{\partial v}(0,0) = 4 \qquad \frac{\partial^2 g}{\partial u \partial v}(0,0) = 3$$

Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(x, y) = g((x + 3)y, x + 2y)$.

Compute the 2^{nd} order Taylor Formula (without remainder) to f at $(0, 0)$.

3. Define $f(x, y) = x^3 - 6xy + 15x + y^2 + 1$.

a) (10 points) Find the second order Taylor expansion of f around the point $(1, 1)$.

b) (15 points) Find all critical points of f, and determine whether they are local maxima, local minima, or saddle points.

Q6. Let $f(x, y) = x^3 + y - xy + 1$.

(a)(5pts.) Find all critical points of f and classsify them. (Local minimum/local maximum/saddle point/degenerate).

(b)(5pts.) Are there points on the curve $y = (x-1)^2$ where ∇f is perpendicular to the curve? If so find them.

(c)(10pts.) Find the absolute maximum and minimum of the function in the region $1 \geq x \geq 0;\ y \geq 0$.

4. (10 points) Find the largest and smallest values attained by the function $f(x, y) = x^3 y$ on the unit circle $\{x^2 + y^2 = 1\}$.

5. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be any C^2 functions.

a) (10 points) Prove the identity

$$\nabla \cdot (\nabla f \times \nabla g) = 0.$$

b) (10 points) Suppose that $\vec{c} : [0, 1] \to \mathbb{R}^3$ is an integral curve (*i.e.*, a flowline) of ∇f such that the initial velocity $\vec{c}\,'(0)$ is nonzero. Prove that, for every $t > 0$, $\vec{c}(t) \neq \vec{c}(0)$ (Hint: Consider $f(\vec{c}(t))$).

3. (10 points) Consider the path of a moving particle $\mathbf{c}(t) = (e^t, t, e^{-t})$,

(a) Find the velocity and acceleration at $t = 1$.

(b) Suppose the particle is set free at $t = 1$. Find the position of the particle at $t = 2$.

6.(15 points) For the given vector field $\mathbf{F}(x, y) = (x, -y)$,

(a) Calculate the divergence and curl of $\mathbf{F}$.

(b) Find the equation of the flow line which passes through $(1, 1)$.

7.(10 points) Let $\mathbf{c}(t)$ be a flow line of a gradient field $\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x})$. Show that $h(t) = V(\mathbf{c}(t))$ is a decreasing function of t.

Q4. (30pts.) Evaluate $\displaystyle\int_0^2 \int_1^{10} \int_{z^2}^4 xz \sin(y^2)\, dy dx dz.$

2. (25pts) Find the absolute maximum values for the function $f(x, y) = xy$ on the rectangle R defined by $-1 \leq x \leq 1, -1 \leq y \leq 1$.

5. (25pts) Let $F(x, y, z) = (3x^2y, x^3 + y^3, 0)$. Find a function f such that $\nabla f = F$. Verify that $\nabla \times F = 0$.

Q3. (a) Find an equation for the plane tangent at $(0, 2, 1/4)$ to the graph of the function $f(x, y) = \frac{e^x}{x^2+y^2}$

(b) Let $g(x, y, z) = x^2 + y^2 - 4z^2$. Find an equation of the plane tangent at $(0, 2, 1/4)$ to a level surface of g.

(c) Describe parametrically the line which is tangent at $(0, 2, 1/4)$ to the curve of intersection between the graph of f and the level surface of g.

Q1. Let $\mathbf{c} : (a, b) \to \mathbb{R}^3$ be a path in three-dimensional space (of class C^2) such that $\mathbf{c}'(t) \neq 0$ for all $t \in (a, b)$. Define a function $\mathbf{k} : (a, b) \to \mathbb{R}^3$

$$\mathbf{k}(t) = -\frac{\mathbf{c}'(t) \times (\mathbf{c}'(t) \times \mathbf{c}''(t))}{\| \mathbf{c}'(t) \|^4}$$

for all $t \in (a, b)$.

Show that if $\| \mathbf{c}'(t) \| = 1$ for all $t \in (a, b)$ then $\mathbf{c}'(t)$ is orthogonal to $\mathbf{c}''(t)$ and $\mathbf{k}(t) = \mathbf{c}''(t)$ for all $t \in (a, b)$.

Q2. Suppose that $\varphi : (a_1, b_1) \to (a, b)$ is a function of class C^2 such that $\varphi'(s) > 0$ for all $s \in (a_1, b_1)$. Define a path $\mathbf{c_1} : (a_1, b_1) \to \mathbb{R}^3$ of class C^2 by $\mathbf{c_1}(s) = \mathbf{c}(\varphi(s))$ for all $s \in (a_1, b_1)$. Here $\mathbf{c}(a, b) \to \mathbb{R}^3$ is the path given in Q1. As in Question 1, let

$$\mathbf{k_1}(s) = -\frac{\mathbf{c}_1'(s) \times (\mathbf{c}_1'(s) \times \mathbf{c}_1''(s))}{\| \mathbf{c}_1'(s) \|^4}$$

Show that $\mathbf{k_1}(s) = \mathbf{k}(\varphi(s))$ for all $s \in (a_1, b_1)$, where $\mathbf{k}(t)$ is the function defined in Problem 1.

(**Note:** You do not need to know how to do problem 1 to do this problem.)

1. *[15 points]*

(a) Let $\mathbf{p}(t)$ be a path and suppose that $\|\mathbf{p}(t)\| = 1$ for all t. Show that $\mathbf{p}'(t)$ is always perpendicular to $\mathbf{p}(t)$.

(b) Let $\mathbf{q}(t)$ be the unit vector parallel to $\mathbf{p}'(t)$ and let $\mathbf{r}(t) = \mathbf{p}(t) \times \mathbf{q}(t)$. Show that $\mathbf{r}'(t) = \mathbf{p}(t) \times \mathbf{q}'(t)$.

(c) Deduce that $\mathbf{r}'(t)$ is a scalar multiple of $\mathbf{p}'(t)$.

4. *[15 points]* A rocket in space follows the path $\mathbf{c}(t) = (2e^t, t, e^{2t})$.

(a) Find the arc length of the rocket's path between the point $(2, 0, 1)$ and the point $(2e, 1, e^2)$.

(b) At time $t = 2$, the rocket's motor is turned off and the rocket is allowed to drift in a straight line. Find the position of the rocket at time $t = 4$.

6. *[25 points]*

(a) Given the function $f(x, y) = 1 - 3x + x^3 + 3xy^2$, find the critical points of f and determine if they are local maxima, local minima, or saddles.

(b) Find the absolute maximum value of the function f in the circle $x^2 + y^2 = \frac{9}{16}$.