110.202 – Calculus III February 28, 2017

Section #5

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Covered: Exam prep, §2.4, §2.5, §2.6

Exam Preparation

- Old Princeton problems on my website.
- 2D limit review – Polar/spherical coordinates:

$$L = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2y + x^2y^3)}{x^2+y^2}$$

$$|\cos^2(\theta)\sin(\theta) + r\cos^2(\theta)\sin^3(\theta)| \le |\cos^2(\theta)\sin(\theta)| + r|\cos^2(\theta)\sin^3(\theta)| \le 1 + r$$
$$\left|r\left(\cos^2(\theta)\sin(\theta) + r\cos^2(\theta)\sin^3(\theta)\right)\right| \le r(1+r)$$

  Cannot use L'Hôpital's rule:

$$L = \lim_{(x,y,z)\to(0,0,0)} \frac{2x^2y\cos(z)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x\frac{2xy}{x^2+y^2}$$

  Since $0 < \frac{|2xy|}{x^2+y^2} \le 1$ when $x, y \neq 0$. Then use squeeze theorem and get $L = 0$.

- Exam topics (nonexhaustive list): Vectors, Matrices/Determinants, Coordinate systems, Limits, Continuity, Derivatives (incl. Chain Rule), Tangent Planes, Paths and Curves, Gradient & Directional Derivative (maybe)
- §2.3 #17 Let P be the tangent plane to the graph of $g(x,y) = 8 - 2x^2 - 3y^2$ at the point $(1, 2, -6)$. let $f(x,y) = 4 - x^2 - y^2$. Find the point on graph of f with tangent plane parallel to P.

  Tangent plane is $z - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0)$.

2.4 – Paths and Curves

HW: 2,5,8,18,24

- Paths in $\mathbb{R}^n$ $c : [a,b] \to \mathbb{R}^n$ and component functions $c(t) = (x(t), y(t), z(t))$.
- Velocity vector $c'(t) = \lim_{h\to 0} \frac{c(t+h) - c(t)}{h}$. Or $c'(t) = (x'(t), y'(t), z'(t))$
- $c'(t)$ is the tangent vector to $c(t)$ at time t, If C is a curve traced out by c and if $c'(t) \neq 0$ then $c'(t)$ is a vector tangent to C at $c(t)$.
- If $c(t)$ is a path and $c'(t_0) \neq 0$ then the tangent line is at the point $c(t_0)$ is $\ell(t) = c(t_0) + (t - t_0)c'(t_0)$

2.5 – Derivative Properties

HW: 3, 5, 6, 8, 12, 18

Chain Rule: Let $x = x(t)$ and $y = y(t)$ be differentiable at t and suppose that $z = f(x, y)$ is differentiable at the point $(x(t), y(t))$. Then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example: Let $z = x^2y - y^2$ where x and y are parametrized as $x = t^2$ and $y = 2t$.

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (2xy)(2t) + (x^2 - 2y)(2) \\
&= (2t^2 \cdot 2t)(2t) + \left((t^2)^2 - 2(2t)\right)(2) \\
&= 8t^4 + 2t^4 - 8t \\
&= 10t^4 - 8t.
\end{aligned}$$

Let $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v) and suppose that $z = f(x, y)$ is differentiable at the point $(x(u, v), y(u, v))$. Then $f(x(u, v), y(u, v))$ has first-order partial derivatives at (u, v) given by

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.
\end{aligned}$$

Let $z = e^{x^2y}$, where $x(u, v) = \sqrt{uv}$ and $y(u, v) = \frac{1}{v}$. Then

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= \left(2xye^{x^2y}\right)\left(\frac{\sqrt{v}}{2\sqrt{u}}\right) + \left(x^2e^{x^2y}\right)(0) \\
&= 2\sqrt{uv} \cdot \frac{1}{v}e^{(\sqrt{uv})^2 \cdot \frac{1}{v}} \cdot \frac{\sqrt{v}}{2\sqrt{u}} + (\sqrt{uv})^2 \cdot e^{(\sqrt{uv})^2 \cdot \frac{1}{v}} \cdot (0) \\
&= e^u + 0 \\
&= e^u \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= \left(2xye^{x^2y}\right)\left(\frac{\sqrt{u}}{2\sqrt{v}}\right) + \left(x^2e^{x^2y}\right)\left(-\frac{1}{v^2}\right) \\
&= 2\sqrt{uv} \cdot \frac{1}{v}e^{(\sqrt{uv})^2 \cdot \frac{1}{v}} \cdot \frac{\sqrt{u}}{2\sqrt{v}} + (\sqrt{uv})^2e^{(\sqrt{uv})^2 \cdot \frac{1}{v}} \cdot \left(-\frac{1}{v^2}\right) \\
&= \frac{u}{v}e^u - \frac{u}{v}e^u \\
&= 0.
\end{aligned}$$

2.6 – Gradient and Directional Derivative

HW: 1, 6, 9, 14, 22

- Directional Derivative
- Direction of Fastest Increase
- Tangent Planes to Level Surfaces
- Gradient Vector Field