110.202 – Calculus III March 27, 2017

Section #7

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Covered: §3.4, §4.1, §4.2

3.4 – Constrained Extrema and Lagrange Multipliers

- When constrained to a surface, a function attains its global maximum and minimum (these are also local max/min). Then, Lagrange Multiplier Thm says if a function has a local maximum or minimum at a point x_0, then there's some real number λ such that $\nabla f(x_0) = \lambda \nabla g(x_0)$. In general, if $\nabla f(x_0) = \lambda \nabla g(x_0)$ is true, then x_0 is a critical point.
- **Example 1:** Find the dimensions of the box with largest volume if the total surface area is *exactly* 64 cm^2.
- **Example 2:** Find the maximum and minimum values of $f(x, y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$. (Note that this is an inequality!)
- **Example 3:** Find the maximum and minimum of $f(x, y, z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2 = 1$.

4.1 – Acceleration and Newton's Second Law

HW: 21, 22

- Recall that if we have a path $\vec{c}(t)$, then its velocity is $\vec{v}(t) = \vec{c}'(t)$.
- Similarly, the velocity is $\vec{a}(t) = \vec{v}'(t) = \vec{c}''(t)$. However, the image of a C^1 path is not always smooth – so $\vec{a}$ is not always defined for a C^1 function.
- We say a differentiable path $\vec{c}$ is regular if at all t, $c'(t) \neq 0$.
- Newton's second law: Net force acting on a particle at a time t is equal to its mass and acceleration at t. Mathematically, this is $\vec{F}(c(t)) = m\vec{a}(t)$
- Circular motion (in 2 dimensions) of mass m at constant speed s in a circle of radius r_0 – $\vec{r}(t) = \left(r_0 \cos\left(\frac{s}{r_0}t\right), r_0 \sin\left(\frac{s}{r_0}t\right)\right)$
- $\frac{s}{r_0}$ is denoted by ω and is called frequency. Then, $\vec{a}(t) = -\omega^2 \vec{r}(t)$ is the acceleration, and $m\vec{a}$ is the centripetal force.

4.2 – Arc Length

HW: 2, 8, 10, 12

- Intuitively, length of a path $\mathbf{c}(t)$ is speed times time, but speed is not constant. Recall speed is $\|\mathbf{c}'(t)\|$. Then, the length of a path $\mathbf{c}(t)$ from time t_0 to t_1 is

$$\int_{t_0}^{t_1} \|\mathbf{c}'(t)\| dt = \int_{t_0}^{t_1} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt.$$

- You can also think about the instantaneous displacement of a particle along a path $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ as

$$d\mathbf{s} = \left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right] dt$$

  and its length is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt.$$

- This arc length differential is useful since it simplifies the arc length to $L = \int_{t_0}^{t_1} d\mathbf{s}$.
- Arc length reparameterization – arc length function $s(t)$ associated with a path $\mathbf{c}(t)$ is

$$s(t) = \int_0^t \|\mathbf{c}'(u)\| du.$$

  Using this, we can reparameterize our path function by arc length s instead of t.

- **Example 4:** Find the length of the curve $\mathbf{c}(t) = (2t, 3\sin(2t), 3\cos(2t))$ on the interval $t \in [0, 2\pi]$.
- **Example 5:** Find the arc length function of the path $\mathbf{c}(t) = (2t, 3\sin(2t), 3\cos(2t))$.
- **Example 6:** What is our position on the curve $\mathbf{c}(t)$ after traveling a distance $\frac{\pi\sqrt{10}}{3}$ from our position at $t = 0$?