110.202 – Calculus III	April 4, 2017

Section #8

TA: David Li

Covered: §4.3, §4.4, §5.1, Exam Review. Practice Problems out Wednesday.

4.3 – Vector Fields

- A vector field in $\mathbb{R}^n$ is a map $\mathbf{F}(x) : A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ that assigns each point $\mathbf{x}$ a vector $\mathbf{F}(\mathbf{x})$.
- Gravitational force at each point is a good example to help with intuition.
- The gradient of a function $\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$ defines a vector field. We call it the **gradient vector field**.
- If $\mathbf{F}$ is a vector field, a flow line is a path $\mathbf{c}'(t)$ such that $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$. Intuitively, it traces the path of a particle in the vector field as a result of the forces described at each point by the vector field.
- Example 1: Show that $\mathbf{c}(t) = (\cos(t), \sin(t), t)$ is a flow line for the vector field $\mathbf{F}(x,y,z) = (-y, x, 1)$.

4.4 – Divergence and Curl

- If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a vector field, the divergence of $\mathbf{F}$ is div $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.
- Interpretation – if $\mathbf{F}$ is a velocity field of a fluid, divergence tells us rate of expansion from the flow of fluid. Can also think about it as a "source" or "sink" – net material entering or leaving that position. Or, think about as "flux density" – flux entering or leaving a point.
- If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a vector field, the curl of $\mathbf{F}$ is curl$l$ $\mathbf{F} = \nabla \times \mathbf{F}$.
- Interpretation – the amount of "rotation" at a specific point. Can also consider the concept of circulation as the amount of "stuff" parallel to the direction of motion. Then, curl is circulation per unit area.
- Other things to know:
  - Gradients are curl-free i.e. $\nabla \times (\nabla f) = 0$. (Expand)
  - Curls are divergence-free i.e. div curl $\mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$. (Expand)
  - Laplace operator ∇^2 defined to be divergence of gradient – $\nabla^2 f = \nabla \cdot (\nabla f)$.
- Example 2: Prove div $(\mathbf{F} + \mathbf{G}) =$ div $\mathbf{F}$ + div $\mathbf{G}$.
- Example 3: Prove curl $(\mathbf{F} + \mathbf{G}) =$ curl $\mathbf{F}$ + curl $\mathbf{G}$.
- Example 4: Prove $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g)$.

5.1 – Introduction to Double Integrals

- Example 5: Evaluate

$$\iint_R x^3 + y^3 dxdy$$

  where $R = [-2, 2] \times [1, 4]$.